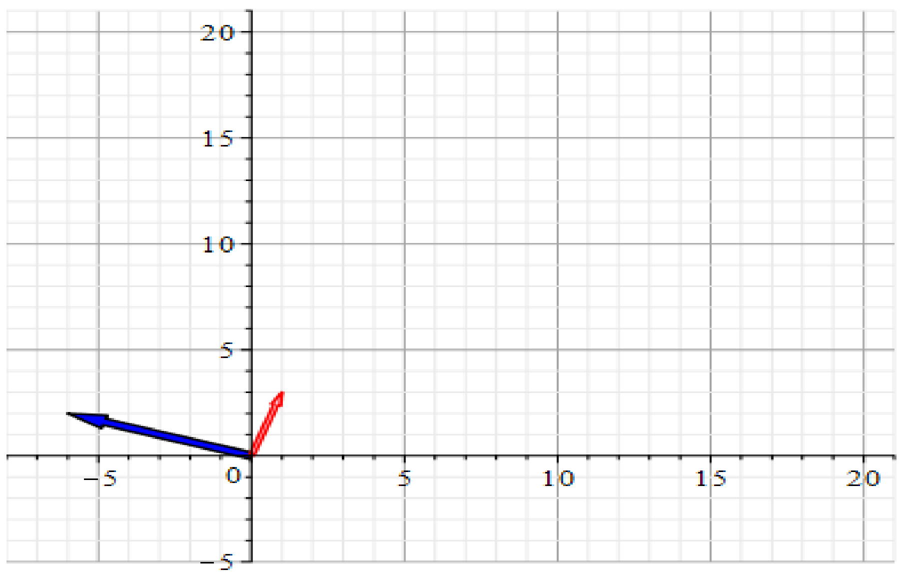
# Spanning

## Definition

A set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$ **spans** $\mathbb{R}^n$ if there are constants $c_1, c_2, c_3, \ldots, c_k$ ($k \geq n$) such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

**Example 1** Do the vectors $u = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$ **span** $\begin{pmatrix} 20 \\ 20 \end{pmatrix}$? If yes, explain how.

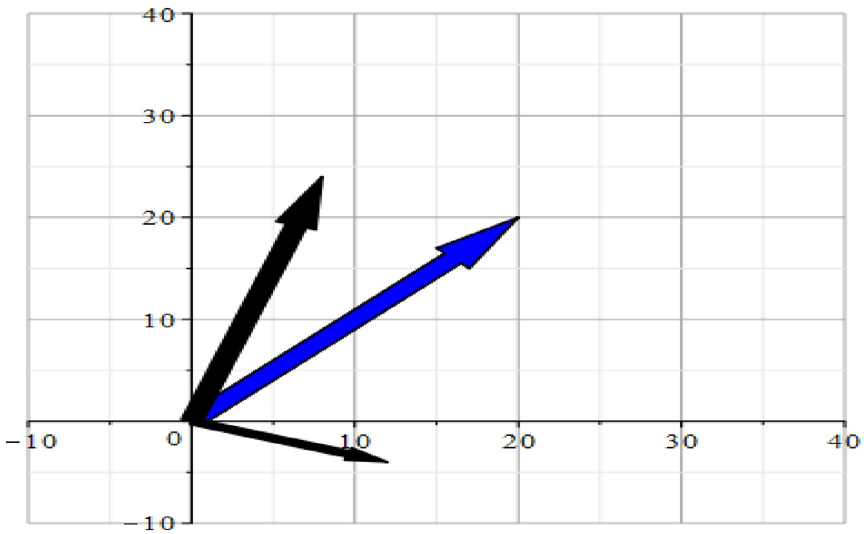



We would be looking for constants $c_1$ and $c_2$ such that

$$c_1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -6 \\ 2 \end{pmatrix} = \begin{pmatrix} 20 \\ 20 \end{pmatrix}$$

$$\begin{bmatrix} 1 & -6 & 20 \\ 3 & 2 & 20 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 20 \\ 0 & 20 & -40 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 20 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -2 \end{bmatrix}$$

**Solution:** $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$




**Example 2**

Can the same vectors $u = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $v = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$ from example 1 **span** the entire $\mathbb{R}^2$? If yes, explain how. We would again be looking for constants $c_1$ and $c_2$ such that

$$c_1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \cdot \begin{pmatrix} -6 \\ 2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{bmatrix} 1 & -6 & x \\ 3 & 2 & y \end{bmatrix} \to \begin{bmatrix} 1 & -6 & x \\ 0 & 20 & y-3x \end{bmatrix} \to \begin{bmatrix} 1 & -6 & x \\ 0 & 1 & \dfrac{y-3x}{20} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6\left(\dfrac{y-3x}{20}\right)+x \\ 0 & 1 & \dfrac{y-3x}{20} \end{bmatrix}$$

**Solution:** $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \dfrac{3y+x}{10} \\ \dfrac{y-3x}{20} \end{pmatrix}$

**Test**

Go back to the point (target) $\begin{pmatrix} 20 \\ 20 \end{pmatrix}$. To span it, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \dfrac{3(20)+20}{10} \\ \dfrac{20-3(20)}{20} \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$, which confirms the result in example 1.

Does it work for any point $\begin{pmatrix} x \\ y \end{pmatrix}$? The answer is yes. Since $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \dfrac{3y+x}{10} \\ \dfrac{y-3x}{20} \end{pmatrix}$,

$$\frac{3y+x}{10}\cdot\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{y-3x}{20}\cdot\begin{pmatrix} -6 \\ 2 \end{pmatrix} = \begin{pmatrix} \dfrac{3y+x}{10} \\ \dfrac{9y+3x}{10} \end{pmatrix} + \begin{pmatrix} \dfrac{18x-6y}{20} \\ \dfrac{2y-6x}{20} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

## Vector Equation Form **( system of linear equations)**

The system

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{array} \qquad \textbf{(1)}$$

can be written in the **vector equation form**

$$x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \qquad \textbf{(2)}$$

## Matrix Equation Form

The system in **(1)** can also be written in the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \tag{3}$$

$$A\vec{x} = \vec{b}$$

**Example**

Express the system

$$\begin{aligned} x + y + z &= 0 \\ y + 2z &= 4 \\ x \quad - \quad z &= 10 \end{aligned}$$

a. in vector equation form;

$$x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 10 \end{bmatrix}$$

b. in matrix equation form;

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 10 \end{bmatrix}$$

**Practice Problem**

$$\begin{aligned} 4a - 2b + c &= 8 \\ a + b + c &= 17 \\ 100a + 10b + c &= -64 \end{aligned}$$

a. Express the system in matrix equation form.

b. Express the system in vector equation form.

c. Solve the system (RREF).

**THEOREM 1**

The system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \quad &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned} \qquad \textbf{(1)}$$

has either a unique solution, no solution, or infinitely many solutions.

**Proof**

Recall that the system in (1) can be express in the form $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

If the system has a unique solution we are done, and the same is true if the system has no solution. We will show that if the system has more than one solution, it must have infinitely many of them. Let us assume that $\vec{x}$ and $\vec{y}$ are both solutions of the system $A\vec{x} = \vec{b}$. Then $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$. Now we let $\vec{z} = t\vec{x} + (1-t)\vec{y}$ where $t$ is any real number. Then

$$A\vec{z} = A(t\vec{x} + (1-t)\vec{y}) = A(t\vec{x} + \vec{y} - t\vec{y}) = A(t\vec{x}) + A(\vec{y}) - A(t\vec{y})$$

$$= tA(\vec{x}) + A(\vec{y}) - tA(\vec{y}) = t\vec{b} + \vec{b} - t\vec{b} = \vec{b}$$

Therefore, $\vec{z}$ is also a solution of the system $A\vec{x} = \vec{b}$, and thus the system has infinitely many solutions.

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## Homework

1. Do vectors $u = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $v = \begin{pmatrix}0\\1\\0\end{pmatrix}$, and $w = \begin{pmatrix}0\\0\\1\end{pmatrix}$ **span** $\mathbb{R}^3$? Explain

2. Do vectors $V_1 = \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$, $V_2 = \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}$ and $V_3 = \begin{bmatrix}1\\-1\\1\\1\end{bmatrix}$ **span** $\mathbb{R}^4$?

3. Can we span $\mathbb{R}^3$ with more than 3 vectors?

4. Can the same vectors $\vec{u} = \begin{pmatrix}1\\0\\-3\end{pmatrix}$, $\vec{v} = \begin{pmatrix}0\\1\\2\end{pmatrix}$ and $\vec{w} = \begin{pmatrix}1\\1\\6\end{pmatrix}$ **span** the entire $\mathbb{R}^3$?

   If yes, explain how.

5. **True or False**

   a. We can span $\mathbb{R}^3$ with 3 vectors as long as no two are on the same line.

   b. We can span $\mathbb{R}^4$ with 4 vectors as long as their RREF results in 4 pivots.

   c. Two vectors $\vec{v}_1$ and $\vec{v}_2$ span $\mathbb{R}^2$ if $\vec{v}_1 \vec{v}_2 \geq 0$.